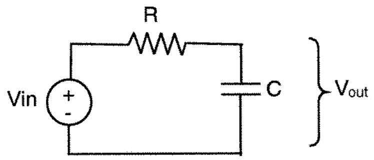
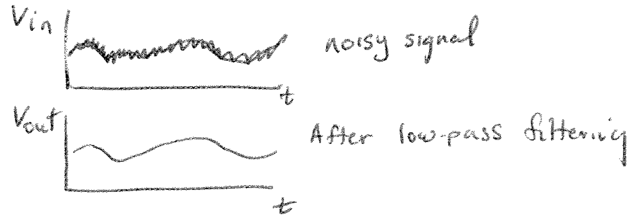
Mechanical Systems Laboratory: Lecture 3

Analysis of a 1st-order, Low-Pass Filter Circuit in the Time and Frequency Domains

The following circuit is a low-pass filter. It is useful to clean up signals with high frequency noise on it:





1. Time Domain Analysis

Let's analyze the response of this circuit to a step input

We'll use the method of undetermined coefficients to solve the differential equation. You can remember this very useful technique for linear, ordinary, differential equations using the following mnemonic:

1. Generals: set the forcing function = 0 and find the general solution to homogenous equation (don't evaluate it's coefficient yet)
2. are Particular: find the particular solution (assume particular soln is same form as forcing function)
3. about Initial Conditions: sum the homogenous and particular solutions and solve for the coefficient to the homogenous equation that satisfies the initial conditions.

KVL: $-V_i + iR + V_o = 0$

$i = C\frac{dV_o}{dt}$

$RC\frac{dV_o}{dt} + V_o = V_i$

$* \quad \frac{dV_o}{dt} = -\frac{1}{RC}V_o + \frac{1}{RC}V_i$

Assume: $V_o(0) = 0$, $V_i = \begin{cases} 0 & t<0 \\ \text{constant} & t>0 \end{cases}$

Homog. $V_h = Ae^{-t/\tau}$ $\quad \tau = RC$

Part: $V_p = V_i$ (assume V_i is a constant)

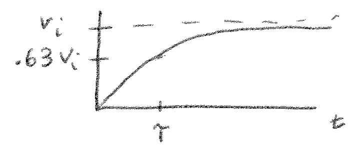
Total: $V_o = Ae^{-t/\tau} + V_i$ but $V(0) = 0 = A + V_i$

$\Rightarrow A = -V_i$

$V_o = V_i(1 - e^{-t/\tau})$

at $t = \tau$

$V_o = V_i(1-e^{-1}) = .63V_i$



Summary of important concepts:

- Method of undetermined coefficients for solving a differential equation.
- Time constant: a 1st order system has gone 63% of the way to its final value after one time constant – standard engineering technique for quantifying "how fast" a system responds.

2. Frequency Domain Analysis

Let's analyze how this system responds to a sinusoidal input. Remember: sine in ⇒ sine out (scaled and shifted), for a linear system. We will use three methods to find the scaling and shifting.

Method 1. Solve differential equation using method of undetermined coefficients (difficult) Assume $V_i = \sin \omega t$

Homogenous solution: $V_h = Ae^{-t/\tau}$ Note: as $t \to \infty$ $V_h \to 0$ ("Transient")

Particular solution: try $V_o = a\sin(\omega t + \phi)$ where a & ϕ are unknown

subst into $*$ = $a\omega\cos(\omega t+\phi) = -\frac{a}{RC}\sin(\omega t+\phi) + \frac{1}{RC}\sin\omega t$

Useful trig. identity: $A\cos(\theta) + B\sin(\theta) = \sqrt{A^2 + B^2}\sin(\theta + \tan^{-1}(\frac{A}{B}))$

$a\omega\cos(\omega t+\phi) + \frac{a}{RC}\sin(\omega t+\phi) = \frac{1}{RC}\sin\omega t = \sqrt{(a\omega)^2 + (\frac{a}{RC})^2}\,\sin(\omega t + \phi + \tan^{-1}\omega RC)$

for the right-most equality to hold

$\frac{1}{RC} = \sqrt{(a\omega)^2 + (\frac{a}{RC})^2} \Rightarrow a = \frac{1}{\sqrt{1+(\omega RC)^2}}$ scaling

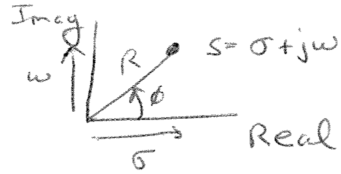
$\phi = -\tan^{-1}\omega RC$ phase shift

$V_o = a\sin(\omega t + \phi)$ with

Method 2: Take Laplace Transform of differential equation that describes circuit, find the transfer function, and solve for frequency response (easier than Method 1)

Brief review of complex variables:
Complex variables keep track of two pieces of information, real and imaginary part, or magnitude and phase

Can think of complex variables as a point in the complex plane.



Can write point in Cartesian or polar coordinates. $s = \sigma + j\omega$

$s = Re^{j\phi}$

To find the magnitude in Cartesian form: $|s| = \sqrt{\sigma^2+\omega^2} = R$

To find the phase in Cartesian form: $\phi_s = \tan^{-1}\frac{\omega}{\sigma} = \phi$

$s = \sigma + j\omega$
$= R\cos\phi + jR\sin\phi$
$= Re^{j\phi}$

Magnitude of two complex variables divided by each other: $\left|\frac{R_1 e^{j\phi_1}}{R_2 e^{j\phi_2}}\right| = \frac{|R_1|}{|R_2|}$

Phase of two complex variables divided by each other: $= \frac{R_1}{R_2} e^{j(\phi_1-\phi_2)}$ $\quad \phi = \phi_1 - \phi_2$

Euler's law $e^{j\phi} = \cos\phi + j\sin\phi$
(can derive by Taylor's expansion)

Now, find the transfer function and frequency response:

$\frac{dV_o}{dt} = \frac{-1}{RC}V_o + \frac{1}{RC}V_i \qquad sV_o = \frac{-1}{RC}V_o + \frac{1}{RC}V_i$

$\frac{V_o}{V_i} = \frac{\frac{1}{RC}}{s+\frac{1}{RC}} = \frac{1}{1+RCs} = G(s)$

$|G(s)|_{s=j\omega} = \frac{|1|}{|1+RCj\omega|} = \frac{1}{\sqrt{1+(RC\omega)^2}}$

$\angle\phi(s)\big|_{s=j\omega} = \tan^{-1}\frac{0}{1} - \tan^{-1}\frac{RC\omega}{1} = -\tan^{-1}RC\omega$

Method 3: Use "impedances" to find transfer function (easiest)

Circuit element	Time domain	Frequency domain	Impedance
Resistor	V(t) = RI(t)	V(s) = R I(s)	R
Capacitor	$V(t) = \frac{1}{C}\int i(t)$	$V(s) = \frac{1}{sC} I(s)$	$\frac{1}{sC}$
Inductor	V(t) = L di/dt	$V(s) = sL\, I(s)$	sL

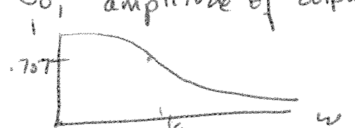
Note: All the usual circuit rules till hold in the frequency domain because of superposition (KVL, KCL, Op amp rules, voltage divider…). So, treat impedances like (frequency dependent) resistors in finding a circuit's transfer function.

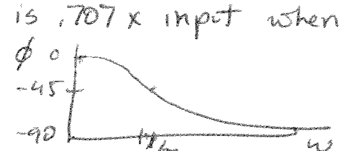
$V_o = \frac{\frac{1}{sC}}{R+\frac{1}{sC}} V_i = \frac{1}{1+RCs} V_i \qquad \frac{V_o}{V_i} = G(s) = \frac{1}{1+RCs}$

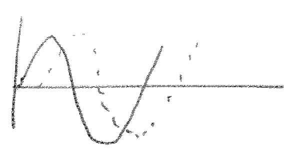
What do the magnitude response (i.e. scaling or attenuation factor) and phase shift response actually look like? Fill in the following chart:

	Magnitude or Scaling	Phase
Small ω	1	0
ω = 1/RC = 1/τ	$\sqrt{\frac{1}{2}} = .707$	-45°
ω ⇒ infinity	0	-90°

So, amplitude of output is .707 x input when $\omega = \frac{1}{\tau}$







The frequency 1/τ is called the "corner frequency" or "bandwidth" of the system. For this low-pass filter, input sinusoids with a frequency higher than the bandwidth are "filtered" or "attenuated".
Summary of important concepts:

- How to find a transfer function and the frequency response
- Impedances
- Corner frequency